From Counting Raindrops to the Fundamental Theorem

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In this workshop you will use the Pröpper-Böhm coproduction `integ()`, a *TI*-92 package, to explore various numerical integration methods, compare their convergence and find a way to the Fundamental Theorem. Although a couple of examples have been provided, additional comments, suggestions, improvements are appreciated by both presentators.
The problems offered are only suggestions and can be changed and extended in many ways. The presentators hope to initialize own ideas and investigations for enriching calculus teaching in the class room. Some of the problems could be a source for class projects.

1) $f(x) = 2(x+2)\,e^{-\frac{x}{3}}$
Find the area enclosed between the graph and the x-axis for $-1 \le x \le 6$.

- Plot the function graph and find an approximate value using 7 strips. Use the *TI*-92 for calculating the function values. Produce a hand sketch of the graph with the in- or circumscribed rectangles or trapeziums.
- Use `integ()` and apply the Monte Carlo Method. (with at least 200 "rain drops").

Add all results of your group and calculate the average.
Produce a graphic representation of the mean for n=200, 400, 600, ... drops.

- Apply other methods using n = 20 strips (graphically), note the results.

Compare the results with the "Random result".
(Geom. Sequence decomposition does not work, because of one negative bound)

- Apply Option ⑥ to compare the convergence of the methods. (Choose 3).
- 1. Generalization: Try three methods using **n** strips. Find the limit for $n \to \infty$.

Do you have any idea, why both LowerSum and UpperSum don´t work?

- 2. Generalization: Try three methods using **a** and **b** as lower and upper bounds. Find again the limit for for $n \to \infty$. Save one result as ***area***.
- Leave integ(). In the Home Screen substitute x for the upper bound b and define an area function ***area(x)*** with x as variable upper bound. Find the 1st derivative of this function!!!

2) $f(x) = \sin(x);$
Find the area between graph and x-axis for $0 \le x \le 2\pi$.
Produce tables for different methods for n = 10,20,30,......... (graphic representation!)
Monte Carlo with n = 100, 200, 300, 400, 500, 600, ...

$f(x) = |\sin(2x)|$ $\quad 0 \le x \le 2\pi$; same as above

3) Explain the "*Pulcherrima*"

4) $f(x) = -\frac{x^3}{15} - \frac{3x^2}{10} + \frac{3x}{2} + \frac{12}{5}$; $-1 \le x \le 3$

- Apply Simpson´s Rule and Pulcherrima using **n** strips. Can you interpret that unexpected result?
- Make it possible to apply the "Geometric Sequence Decomposition" despite the negative lower bound!
-

5) Find the generalized integration rules for:

$$\int_a^b c\, x^p\ dx; \qquad \int_a^b c\, e^{p\,x}\ dx; \qquad \int_a^b c \sin x\ dx; \qquad \int_a^b \frac{c}{x}\ dx$$

6) Make a conjecture for a formula for the area of an ellipse.

7) How to handle a discontinuity within the integration range?

- a) $\int_0^4 \frac{10}{\sqrt[3]{x^2}}\,dx$ with n = 10,......,60 (10) any method!?!

  which of them will not work and why not?

$\int_0^4 \frac{10}{\sqrt[3]{x^2}}\,dx$ with **n** strips and n → ∞, a → 0

- b) $\int_0^4 \frac{10}{x^2}\,dx$ with n = 10,........., 60 (10) any method!?!
- Can you see a difference in the convergence behaviour?
- $\int_0^4 \frac{10}{x^2}\,dx$ with **n** strips and n → ∞, a → 0

8) Find a function with two jump discontinuities. Produce a numerical approximation for the area between the function graph and y = 0. Is there any thing you have to take care of? Plot the function graph and the integral function on the same set of axes.

9) Find a function with a non differentiable place within the integration range. Plot the integral function.

10) There are some relations between the various numerical approximations. Verify - and then proof!? - the following ones using a self chosen function:

well known and easy explained:	$2 * S_{TRAP} = S_{LEFT} + S_{RIGHT}$.
well known?	$3 * S_{SIMPSON} = S_{TRAP} + 2 * S_{MID}$
also well known?	3*SIMPSON(h) = 4*TRAPEZ(2h) - TRAPEZ(h)

11) Try the following: take a generic function g(x) or wolfgang(x) or josef(x), bounds a and b, number of strips = 1 and choose any method. Interpret the result.

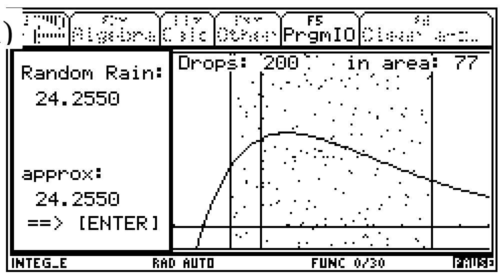
12) Find a numerical approximation for $\displaystyle\int_2^{10} \frac{5(e^{\frac{x}{2}} + x^2)}{10 + x^3}\, dx$

How many strips are necessary to obtain a 4 digit accuracy

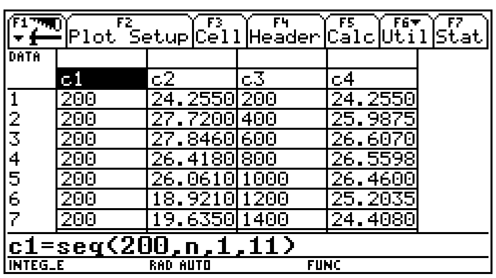
- Left- or RightSum,
- Midpoint Rule, Trapezium Sum,
- Simpson´s Rule, Pulcherrima.
- Compare with the *TI*-92´s `nInt`-function.

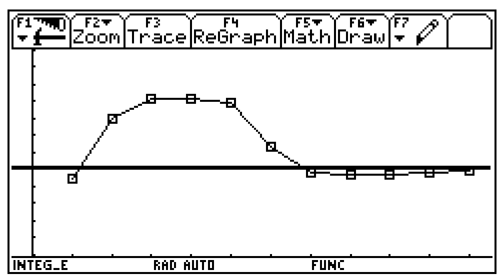
13) Find a numerical approximation for $\displaystyle\int_2^{4} \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\left(\frac{X-\overline{X}}{\sigma}\right)^2}\, dX\,;\quad \overline{X} = 3.5,\ \sigma = 1.5$

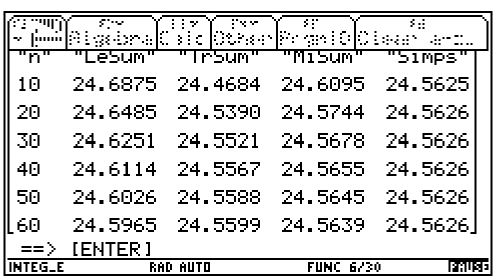
Selected answers

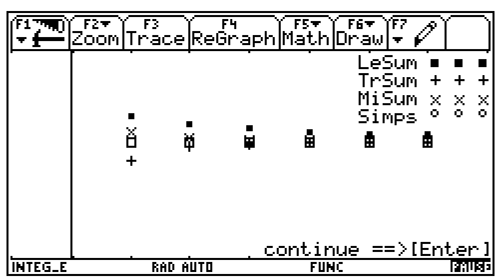
1) 

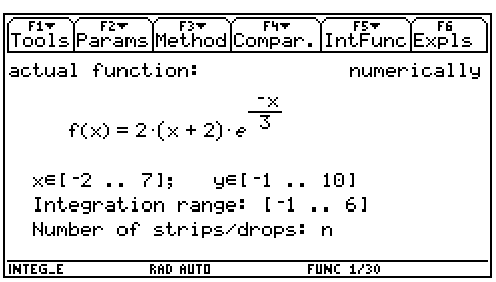
Area of the rectangle = 7 * 9 = 63
63 : area = 77 : 200 --> area = 24.2550

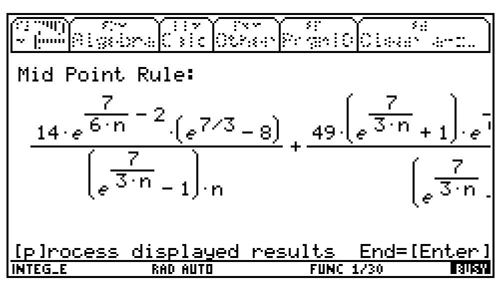


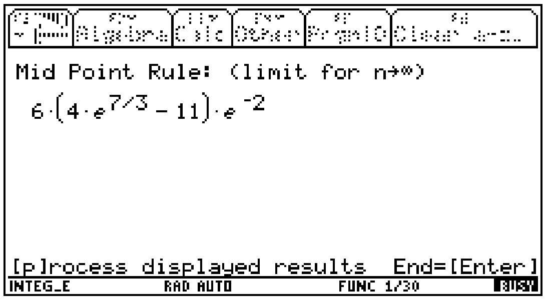


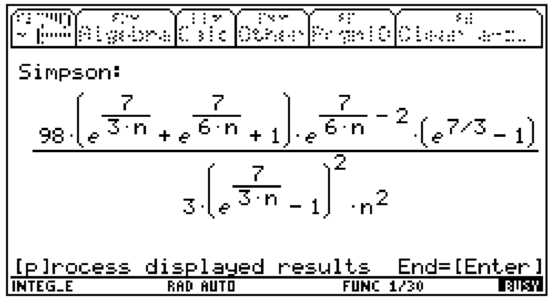


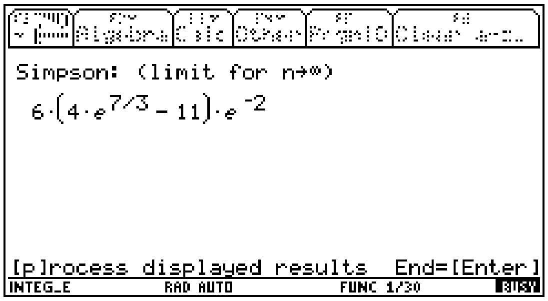


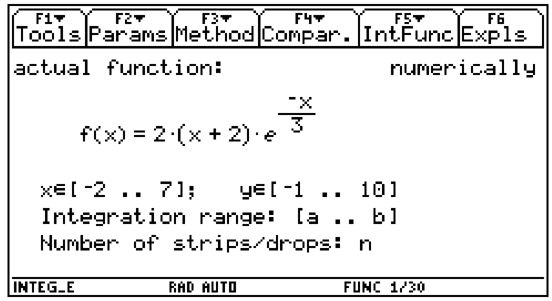


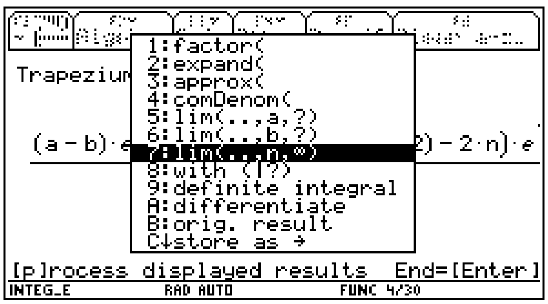


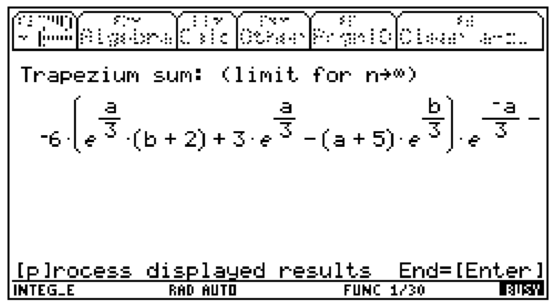




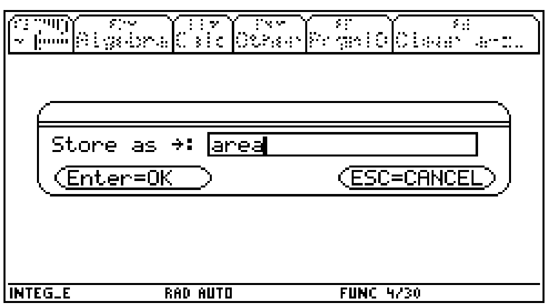


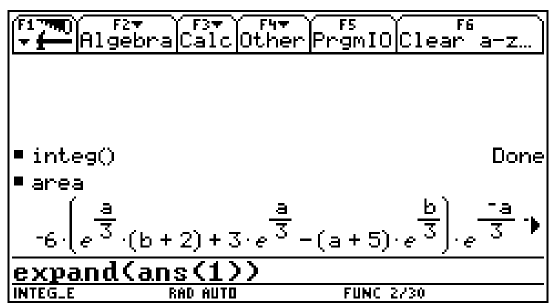


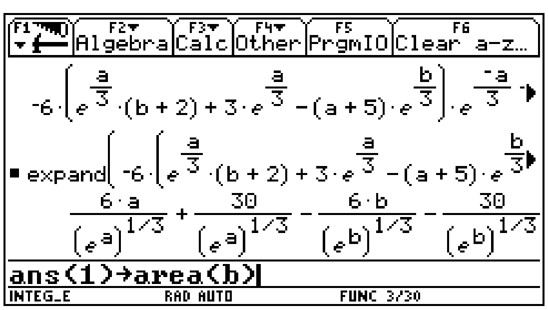


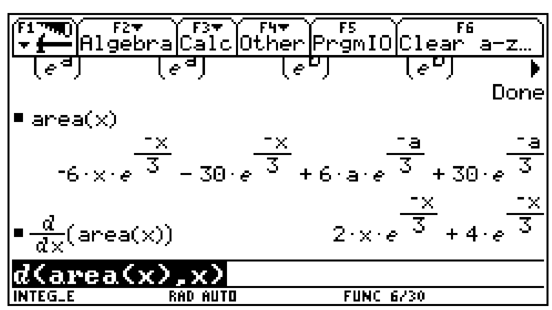


Find an area function and produce its first derivative.

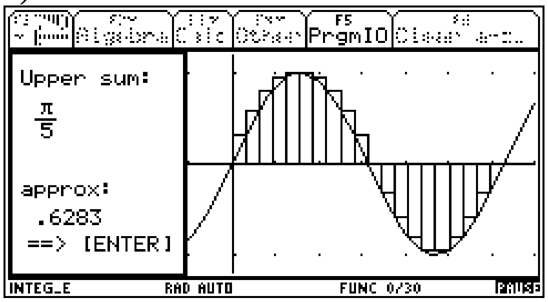




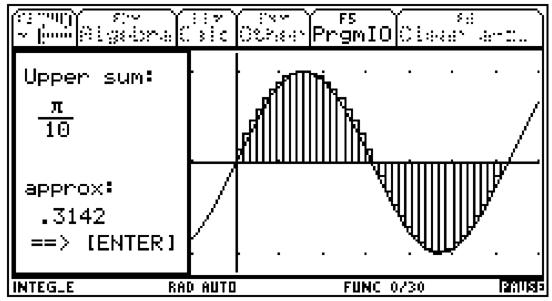




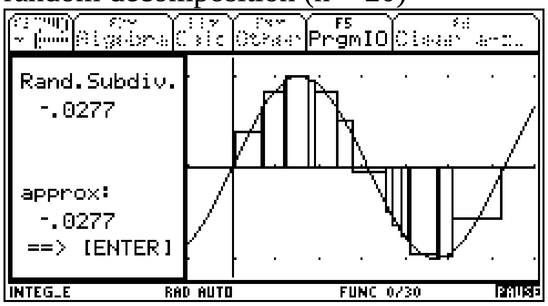
2) n = 20



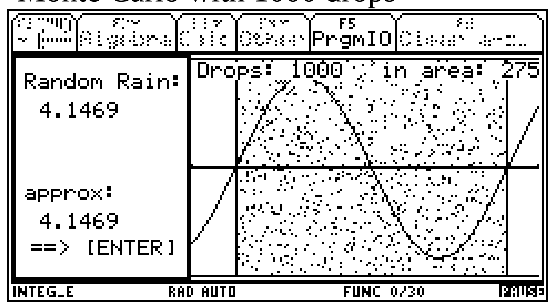
n = 40



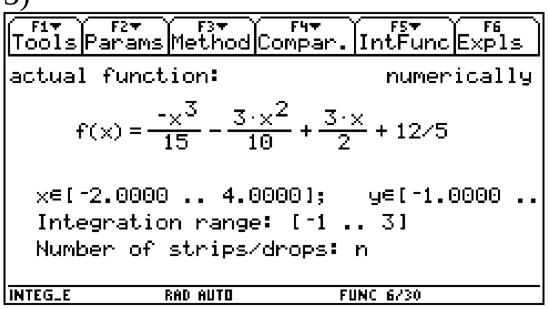
random decomposition (n = 20)

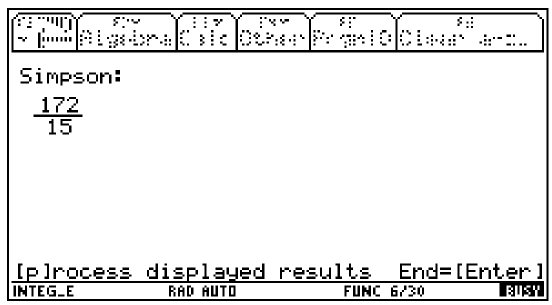


Monte Carlo with 1000 drops

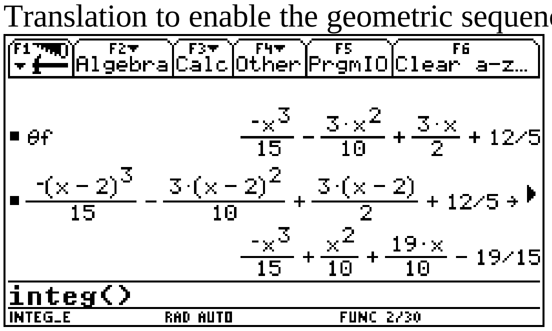


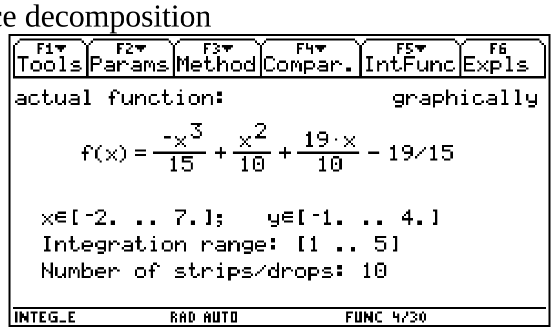
3)

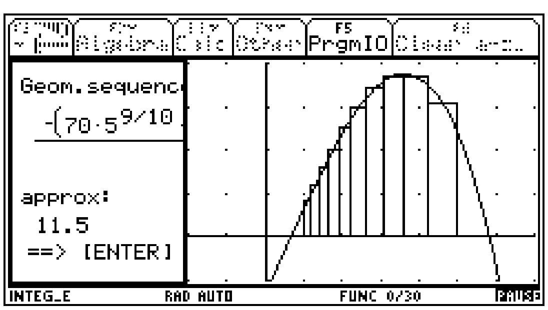




Translation to enable the geometric sequence decomposition

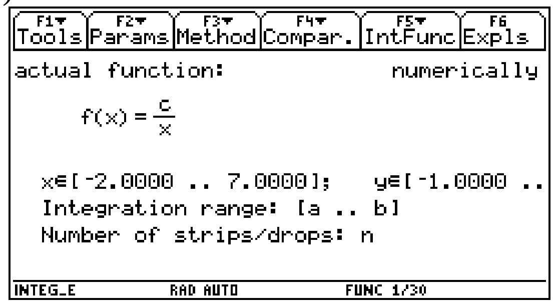


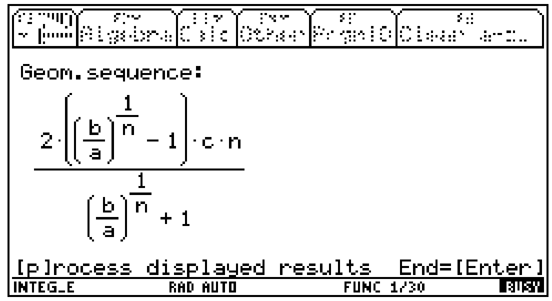


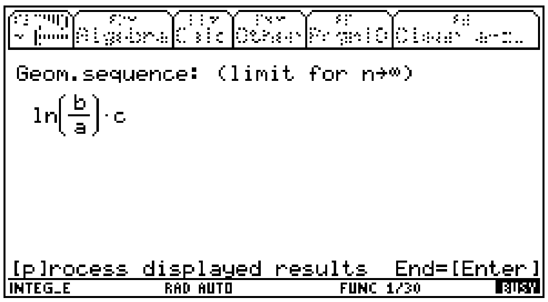


The generalization with limit $n \to \infty$ does not work. The *TI* is unable to compute the sum. But you can do it by parts: -x^3/15 leads to part1, ...
The sum of the four limits is: 172/15.
(fortunately!!!)

5)

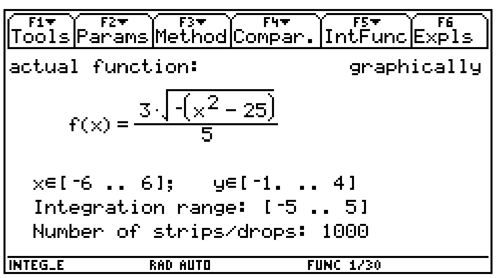


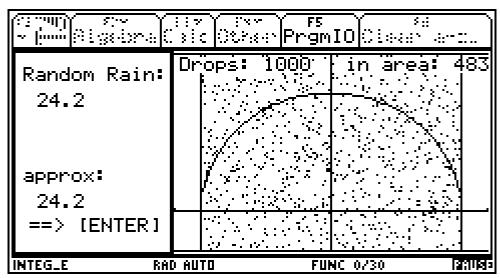


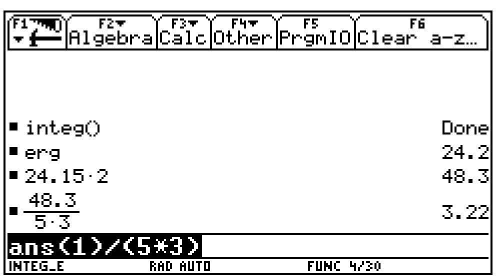


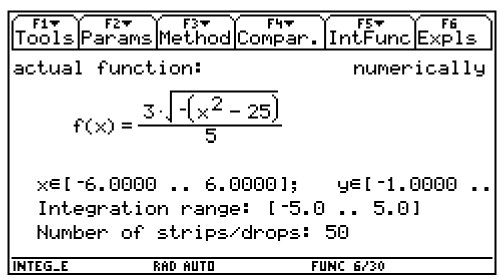
Unfortunately the *TI* is unable to perform the summation for the trig func-
tions. So we cannot derive the integration rules for sin(x) and cos(x) - unlike carrying through the same process with *DERIVE*.

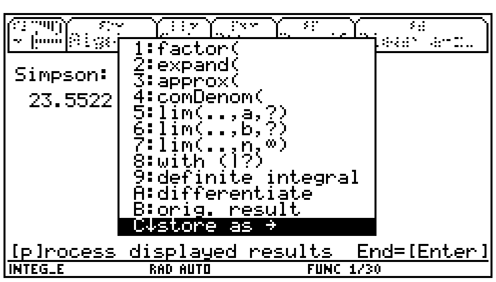
6) Ellipse (a = 5, b = 3): $\frac{x^2}{25} + \frac{y^2}{9} = 1 \rightarrow y = \pm\frac{3}{5}\sqrt{25 - x^2}$

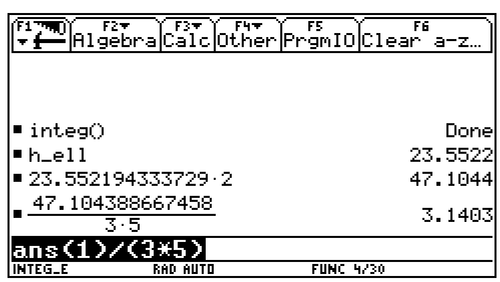






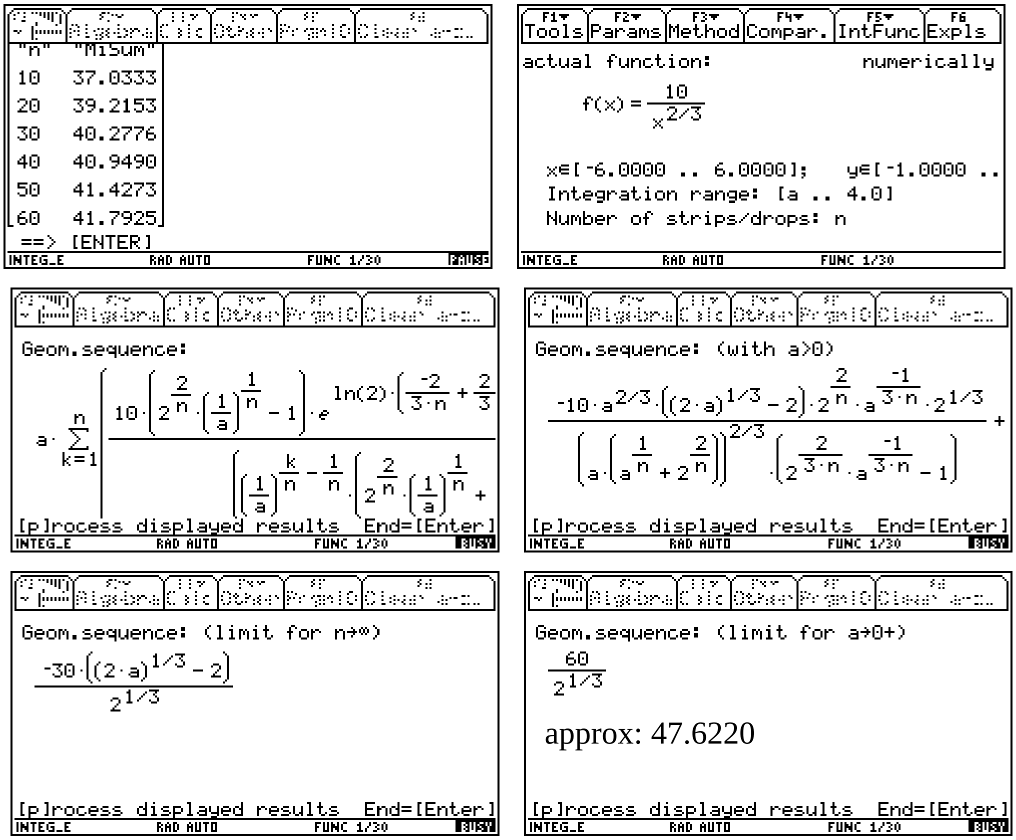




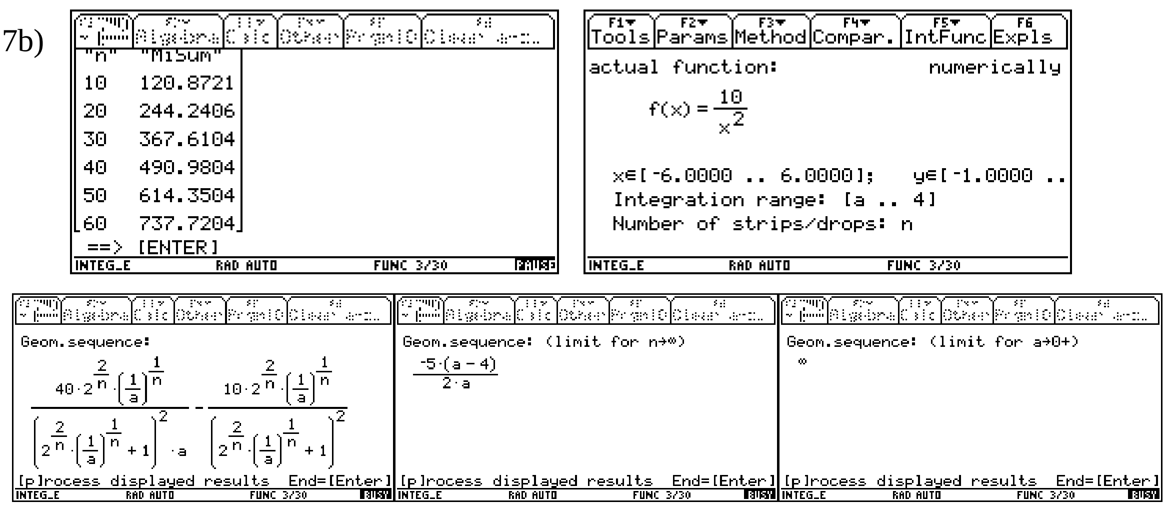


The root describes a circle with area = 25π, all function values are multilpied by 3/5, so the area of the ellipse should be: ....

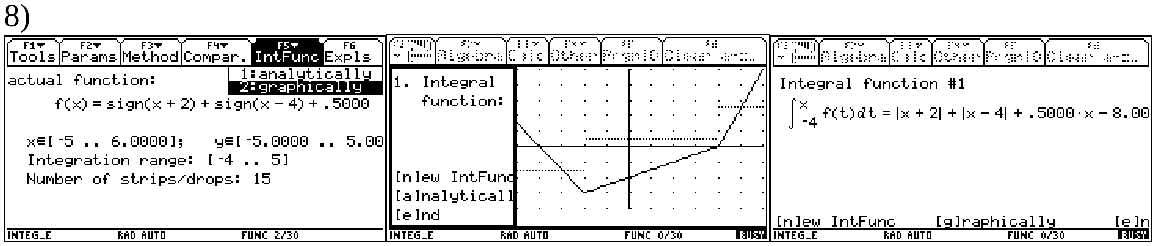
7a)


"n" "MiSum"
10 37.0333
20 39.2153
30 40.2776
40 40.9490
50 41.4273
60 41.7925
==> [ENTER]
Tools Params Method Compar. IntFunc Expls
actual function: numerically
f(x) = 10/x^(2/3)
x∈[-6.0000 .. 6.0000]; y∈[-1.0000 ..
Integration range: [a .. 4.0]
Number of strips/drops: n
Geom.sequence:
Geom.sequence: (with a>0)
Geom.sequence: (limit for n→∞)
-30·((2·a)^(1/3) - 2) / 2^(1/3)
Geom.sequence: (limit for a→0+)
60 / 2^(1/3)
approx: 47.6220
[p]rocess displayed results End=[Enter]

7b)


"n" "MiSum"
10 120.8721
20 244.2406
30 367.6104
40 490.9804
50 614.3504
60 737.7204
==> [ENTER]
Tools Params Method Compar. IntFunc Expls
actual function: numerically
f(x) = 10/x^2
x∈[-6.0000 .. 6.0000]; y∈[-1.0000 ..
Integration range: [a .. 4]
Number of strips/drops: n
Geom.sequence:
Geom.sequence: (limit for n→∞)
-5·(a - 4) / 2·a
Geom.sequence: (limit for a→0+)
∞
[p]rocess displayed results End=[Enter]

8)


Tools Params Method Compar. IntFunc Expls
actual function:
1:analytically
2:graphically
f(x) = sign(x + 2) + sign(x - 4) + .5000
x∈[-5 .. 6.0000]; y∈[-5.0000 .. 5.00
Integration range: [-4 .. 5]
Number of strips/drops: 15
1. Integral function:
[n]ew IntFunc
[a]nalytical
[e]nd
Integral function #1
∫_{-4}^x f(t)dt = |x + 2| + |x - 4| + .5000·x - 8.00
[n]ew IntFunc [g]raphically [e]n